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Optimal Harvesting of a Renewable Economic Resource in a Model with Bertalanffy Growth Law I

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Abstract. We examine the generalized Bertalanffy growth model as a basis of bioeconomic models for renewable resource exploitation. We show that the model has a built-in protective mechanism that may be utilized to enhance conservation of the resource. Its application to the management of open-access renewable resources shows a much stronger economic condition than in the logistic one which is needed for biological overexploitation to occur.

1. INTRODUCTION

Following Bertalanffy [1], the equation

$$\dot{x} = \alpha x^m - \beta x^n \tag{1.1}$$

will be assumed to describe the growth of population biomass of a renewable resource over time, where $x(t)$ is the aggregate biomass of the resource population. By introducing the new variables

$$y = \frac{x}{\hat{x}}, \quad \tau = \beta t, \quad \hat{x} = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{n-m}} \tag{1.2}$$

and assuming $n = 1$, equation (1.1) assumes the form

$$\dot{y} = y^m - y, \tag{1.3}$$

without any loss of generality.

2. HARVESTING MODELS

When the biomass system described by equation (1.3) is subject to exploitation, the net growth equation is given by

$$\dot{y} = y^m - y - h(t) \tag{2.1}$$

where $h(t) > 0$ is the rate of removal of biomass at time t in appropriate units.

<u>ResultI</u>: When sustainable yield harvesting is used, the maximum sustainable yield occurs at the level of biomass $y_{MSY} = m^{\frac{1}{(1-m)}}$ with the property:

$$\lim_{m \to 1} y_{MSY} = e^{-1} \qquad \text{(where } e \text{ is Euler's constant)}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX and the maximum sustainable yield is given by:

$$h_{MSY} = (1-m)m^{\frac{m}{(1-m)}} \quad \text{with the property} \quad \lim_{m\to 1} h_{MSY} = 0$$

Result II: When density-dependent harvesting is used, a unique nonzero equilibrium solution

$$y^{\star}(m;E) = \left[\frac{1}{1+E}\right]^{\frac{1}{(1-m)}} \tag{2.2}$$

exists which is a decreasing function of both effort level (E) and the population parameter (m). The corresponding equilibrium yield

$$Y(m;E) = E\left[\frac{1}{1+E}\right]^{\frac{1}{(1-m)}} \tag{2.3}$$

is a nonsymmetric increasing function of the harvesting effort and a decreasing function of the population parameter (m).

Result III: An inverse relationship exists between MSY effort (E_{MSY}) and the population parameter (m), namely

$$E_{MSY} = \frac{1-m}{m} \tag{2.4}$$

The knowledge that a population under exploitation has a small (close to zero) or large (close to 1) m value automatically determines the range of harvesting intensity that could be applied in harvesting the population in order to enhance conservation. Thus the Bertalanffy model, unlike any other standard model in the literature, possesses a built-in protective mechanism for the resource under exploitation.

Result IV: The characteristic return time for the unharvested and harvested population are increasing functions of the population parameter (m)

$$CRT|_{E=0} = CRT(0) = \frac{1}{1-m} \tag{2.5}$$

$$CRT(E) = \frac{1}{(1-m)(1+E)} \tag{2.6}$$

$$\frac{CRT(E)}{CRT(0)} = \frac{1}{1+E} \tag{2.7}$$

Thus for a population biomass system whose growth follows the Bertalanffy growth law, its characteristic return time decreases steadily as the harvesting effort increases. At the MSY level, the characteristic return time for the harvested population is only m times as long as for the unharvested population. Thus the stability of a population following the Bertalanffy growth model can be enhanced through harvesting.

Result V: In the management of open-access renewable resources where the goal is the maximization of the present value of economic rent and the Bertalanffy model is the appropriate one, biological overexploitation occurs whenever price-to-cost ratio exceeds Euler's constant. This is a more conservative result than that obtained by Clark [2] who utilized the logistic model in his work.

3. Maximizing Economic Rent

In this section, we discuss the problem of finding the aggregate biomass level that maximizes economic rent.

$$\max_x R(x) = [p(h) - C(x,h)]h \tag{3.1}$$

It is assumed that

$$\begin{aligned} p(h) &= \frac{p}{h^{1-\alpha}}, \qquad 0 < \alpha < 1 \\ C(x,h) &= \lambda + \frac{c}{h} \end{aligned} \tag{3.2}$$

where c and p are unit cost and price respectively and λ is the fixed unit cost of harvesting. Under a sustainable yield harvesting policy,

$$h = f(x) = x^m - x \tag{3.3}$$

In this case, the economic rent is

$$R(x) = p[f(x)]^\alpha - \lambda f(x) - c \tag{3.4}$$

In the special case where $m = \frac{1}{2}$, the solutions are:

$$\begin{aligned} x_1^\star &= \frac{1}{2}[(1-2k) - (1-4k)^{\frac{1}{2}}] \\ x_2^\star &= \frac{1}{4} = x_{MSY} \\ x_3^\star &= \frac{1}{2}[(1-2k) + (1-4k)^{\frac{1}{2}}] \end{aligned} \tag{3.5}$$

where

$$k = \left(\frac{\alpha p}{\lambda}\right)^{\frac{1}{1-\alpha}} \tag{3.6}$$

Result VI: If the fixed cost-to-price ratio $\frac{\lambda}{p} \leq \alpha 4^{1-\alpha}$ $(k \geq \frac{1}{4})$, $x_2^\star = \frac{1}{4}$ is both the level of maximum biological and economic yield.

Result VII: If the fixed cost-to-price ratio $\frac{\lambda}{p} > \alpha 4^{1-\alpha}$ $(k > \frac{1}{4})$, there are three extremum points. In this case, $x_2^\star = \frac{1}{4}$ yields a minimum value for the economic rent and the continuity of $R(x)$ indicates that $x_1^\star$ and $x_3^\star$ each gives a maximum value for $R(x)$, since $x_1^\star < x_2^\star < x_3^\star$.

Result VIII: If the unit cost of harvesting is a function only of the biomass level, a maximizing biomass level greater than the level of maximum sustainable yield is obtained. This result is in accord with Clark [4] and Colbert [5], both of whom worked with the logistic model of growth.

Result IX: If density-dependent harvesting policy is followed, the economic rent is

$$R(x) = pE^{\alpha}x^{\alpha} - \lambda Ex - c. \tag{3.7}$$

If the economic condition

$$\left(\frac{\alpha p}{\lambda}\right)^{\frac{1}{1-\alpha}} \geq \frac{1}{4} \tag{3.8}$$

holds, density-dependent harvesting is not only a rational policy, but both biologically and economically superior to sustainable yield policy.

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